Module – III
Probability Distributions

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Probability Distributions

In a practical situation, one may be interested in finding the probabilities of all the events and may wishes to have the results in a tabular form for any future reference. Since for an experiment having **n** outcomes, totally, there are 2^n totally events; finding probabilities of each of these and keeping them in a tabular form may be an interesting problem.

Thus, if we develop a procedure, using which if it is possible to compute the probability of all the events, is certainly an improvement. The aim of this chapter is to initiate a discussion on the above.

Also, in many random experiments, outcomes may not involve a numerical value. In such a situation, to employ mathematical treatment, there is a need to bring in numbers into the problem. Further, probability theory must be supported and supplemented by other concepts to make application oriented. In many problems, we usually do not show interest on finding the chance of occurrence of an event, but, rather we work on an experiment with lot of expectations

Considering these in view, the present chapter is dedicated to a discussion of random variables which will address these problems.

<u>First what is a random variable?</u>

Let S denote the sample space of a random experiment. A random variable means it is a rule which assigns a numerical value to each and every outcome of the experiment. Thus, random variable may be viewed as a function from the sample space S to the set of all real numbers; denoted as $f:S \rightarrow R$. For example, consider the random experiment of tossing three fair coins up. Then S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}. Define f as the number of heads that appear. Hence, $f(HHH)=3$, $f(HHT)=2$, $f(HTH)=2$, $f(THH)=2$, $f(HTT)=1$, $f(THT)=1$, $f(TTH)=1$ and $f(TTT)=0$. The same can be explained by means of a table as given below:

HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
3	2	2	2	1	1	1	1

Note that all the outcomes of the experiment are associated with a unique number. Therefore, f is an example of a random variable. Usually a random variable is denoted by using upper case letters such as X, Y, Z etc. The image set of the random variable may be written as $f(S)=\{0, 1, 2, 3\}$.

A random variable is divided into

- **Discrete Random Variable (DRV)**
- **Continuous Random Variable (CRV)**.

If the image set, **X(S),** is either finite or countable, then X is called as a discrete random variable, otherwise, it is referred to as a continuous random variable i.e. if X is a CRV, then X(S) is infinite and un – countable.

<u>Example of Discrete Random Variables:</u>

1. In the experiment of throwing a die, define X as the number that is obtained. Then X takes any of the values 1 – 6. Thus, X(S) = {1, 2, 3. . . 6} which is a finite set and hence X is a DRV.
2. Let X denotes the number of attempts required for an engineering graduate to obtain a satisfactory job in a firm? Then X(S) = {1, 2, 3,. . . . }. Clearly X is a DRV but having a image set countably infinite.
3. (iii) If X denote the random variable equals to the number of marks scored by a student in a subject of an examination, then X(S) = {0, 1, 2, 3,................. 100}. Thus, X is a DRV, Discrete Random Variable.
4. (iv) In an experiment, if the results turned to be a subset of the non – zero integers, Then it may be treated as a Discrete Random Variable.

Examples of Continuous Random Variable:

1. Let X denote the random variable equals the speed of a moving car, say, from a destination A to another location B, then it is known that speedometer indicates the speed of the car continuously over a range from 0 up to 160 KM per hour. Therefore, X is a CRV, Continuously Varying Random Variable.
2. Let X denotes the monitoring index of a patient admitted in ICU in a good hospital. Then it is a known fact that patient's condition will be watched by the doctors continuously over a range of time. Thus, X is a CRV.

3. Let X denote the number of minutes a person has to wait at a bus stop in Bangalore to catch a bus, then it is true that the person has to wait anywhere from 0 up to 20 minutes (say). Will you agree with me? Since waiting to be done continuously, random variable in this case is called as CRV.
4. Results of any experiments accompanied by continuous changes at random over a range of values may be classified as a continuous random variable.

Probability function/probability mass function $f(x_i) = P[X = x_i]$ of a discrete random variable:

Let X be a random variable taking the values, say $X: \quad x_1 \quad x_2 \quad x_3 \; . \; . \; . \; x_n$ then $f(x_i) = P[X = x_i]$ is called as probability mass function or just probability function of the discrete random variable, X. Usually, this is described in a tabular form:

$X = x_i$	x_1	x_2	x_3	.	.	.	x_n
$f(x_i)$	$P(2 \leq X < 5)$ $f(x_1)$	$f(x_2)$	$f(x_3)$	.	.	.	$f(x_n)$

Note: When X is a discrete random variable, it is necessary to compute $f(x_i) = P[X = x_i]$ for each i = 1, 2, 3 . . n. This function has the following properties:

- $f(x_i) \geq 0$

- $0 \le f(x_i) \le 1$
- $\sum_i f(x_i) = 1$

On the other hand, X is a continuous random variable, then its probability function will be usually given or has a closed form, given as $f(x) = P(X = x)$ where x is defined over a range of values., it is called as probability density function usually has some standard form. This function too has the following properties:

- $f(x) \ge 0$
- $0 \le f(x) \le 1$
- $\int_{-\infty}^{\infty} f(x) = 1$.

To begin with we shall discuss in detail, discrete random variables and its distribution functions. Consider a discrete random variable, **X** with the distribution function as given below:

$X = x_i$	x_1	x_2	x_3	.	.	.	x_n
$f(x_i)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	.	.	.	$f(x_n)$

Using this table, one can find probability of various events associated with X. For example,

- $P(x_i \le X \le x_j) = P(X = x_i) + P(X = x_{i+1}) +$ up to $+ P(X = x_j)$

  $= f(x_i) + f(x_{i+1}) + f(x_{i+2}) +$ up to $+ f(x_{j-1}) + f(x_j)$

- $P(x_i < X < x_j) = P(X = x_{i+1}) + P(X = x_{i+2}) + . \; . \; . + P(X = x_{j-1})$

  $= f(x_{i+1}) + f(x_{i+2}) +$ up to $+ f(x_{j-1})$

- $P\left(X > x_j\right) = 1 - P\left(X \le x_{j-1}\right)$

  $= 1 - \left\{P\left(X = x_1\right) + P\left(X = x_2\right) \text{ up to } + P\left(X = x_{j\text{-}1}\right)\right\}$

The probability distribution function or cumulative distribution function is given as

$$F(x_t) = P(X \le x_t) = P\left(X = x_1\right) + P\left(X = x_2\right) + \text{ up to } + P\left(X = x_t\right)$$

It has the following properties:

- $F(x) \ge 0$
- $0 \le F(x) \le 1$
- When $x_i < x_j$ then $F\left(x_i\right) < F\left(x_j\right)$ i.e. it is a strictly monotonic increasing function.
- when $x \to \infty$, $F(x)$ approaches 1
- when $x \to -\infty$, $F(x)$ approaches 0

<u>A brief note on Expectation, Variance, Standard Deviation of a Discrete Random Variable:</u>

- $E(X) = \sum_{i=1}^{i=n} x_i \cdot f\left(x_i\right)$
- $E\left(X^2\right) = \sum_{i=1}^{i=n} x_i^2 \cdot f\left(x_i\right)$
- $\text{Var}(X) = E\left(X^2\right) - \left\{E(X)\right\}^2$

<u>ILLUSTRATIVE EXAMPLES:</u>

1. The probability density function of a discrete random variable X is given below:

X:	0	1	2	3	4	5	6
$f(x_i)$:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find (i) k; (ii) $F(4)$; (iii) $P(X \geq 5)$; (iv) $P(2 \leq X < 5)$ (v) E(X) and (vi) Var (X).

Solution: To find the value of k, consider the sum of all the probabilities which equals to 49k. Equating this to 1, we obtain $k = \frac{1}{49}$. Therefore, distribution of X may now be written as

X:	0	1	2	3	4	5	6
$f(x_i)$:	$\frac{1}{49}$	$\frac{3}{49}$	$\frac{5}{49}$	$\frac{7}{49}$	$\frac{9}{49}$	$\frac{11}{49}$	$\frac{13}{49}$

Using this, we may solve the other problems in hand.

$$F(4) = P[X \leq 4] = P[X = 0] + P[X = 1] + P[X = 2] + P[X = 3] + P[X = 4] = \frac{25}{49}.$$

$$P[X \geq 5] = P[X = 5] + P[X = 6] = \frac{24}{49}$$

$P[2 \leq X < 5] = P[X = 2] + P[X = 3] + P[X = 4] = \frac{21}{49}$. Next to find E(X), consider

$E(X) = \sum_i x_i \cdot f(x_i) = \frac{203}{49}$. To obtain Variance, it is necessary to compute

$E(X^2) = \sum_i x_i^2 \cdot f(x_i) = \frac{973}{49}$. Thus, Variance of X is obtained by using the relation, $\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 = \frac{973}{49} - \left(\frac{203}{49}\right)^2$.

2. A random variable, X, has the following distribution function.

X :	-2	-1	0	1	2	3
$f(x_i)$:	0.1	k	0.2	$2k$	0.3	k

Find (i) k, (ii) $F(2)$, (iii) $P(-2 < X < 2)$, (iv) $P(-1 < X \le 2)$, (v) E(X) , Variance.

Solution: Consider the result, namely, sum of all the probabilities equals 1,

$0.1 + k + 0.2 + 2k + 0.3 + k = 1$ Yields k = 0.1. In view of this, distribution function of X may be formulated as

X :	-2	-1	0	1	2	3
$f(x_i)$:	0.1	0.1	0.2	0.2	0.3	0.1

Note that $F(2) = P[X \le 2] = P[X = -2] + P[X = -1] + P[X = 0] + P[X = 1] + P[X = 2]$

$= 0.9$. The same also be obtained using the result,

$F(2) = P[X \le 2] = 1 - P[X < 1] = 1 - \{P[X = -2] + P[X = -1] + P[X = 0]\}$ **= 0.6.**

Next, $P(-2 < X < 2) = P[X = -1] + P[X = 0] + P[X = 1] = 0.5$ **.**

Clearly, $P(-1 < X \le 2)$ **= 0.7. Now, consider** $E(X) = \sum_i x_i \cdot f(x_i)$ **= 0.8.**

Then $E(X^2) = \sum_i x_i^2 \cdot f(x_i)$ **= 2.8.** $\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2$ **= 2.8 - 0.64 = 2.16.**

<u>A DISCUSSION ON A CONTINUOUS RANDOM VARIABLE</u>

<u>AND IT'S DENSITY FUNCTION:</u>

Consider a continuous random variable, X. Then its probability density is usually given in the form of a function $f(x)$ with the following properties.

(i) $f(x) \ge 0$, (ii) $0 \le f(x) \le 1$ and (*iii*) $\int_{-\infty}^{\infty} f(x)\, dx = 1$ **.**

Using the definition of $f(x)$, it is possible to compute the probabilities of various events associated with **X**.

- $P(a \le X \le b) = \int_a^b f(x)\, dx$, $\quad P(a < X \le b) = \int_a^b f(x)\, dx$
- $P(a < X < b) = \int_a^b f(x)\, dx$, $\quad F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$
- $E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$, $\quad E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx$
- $\mathrm{Var}(X) = E\left(X^2\right) - \{E(X)\}^2$
- $P(a < X < b) = F(b) - F(a)$
- $f(x) = \dfrac{dF(x)}{dx}$, **if the derivative exists**

<u>SOME STANDARD DISTRIBUTIONS OF A DISCRETE RANDOM VARIABLE:</u>

Binomial distribution function: Consider a random experiment having only two outcomes, say success (**S**) and failure (**F**). Suppose that trial is conducted, say, n number of times. One might be interested in knowing how many number of times success was achieved. Let **p** denotes the probability of obtaining a success in a single trial and **q** stands for the chance of getting a failure in one attempt implying that **p + q = 1**. If the experiment has the following characteristics;

- the probability of obtaining failure or success is same for each and every trial
- trials are independent of one another
- probability of having a success is a finite number, then

We say that the problem is based on the binomial distribution. In a problem like this, we define **X** as the random variable equals the number of successes obtained in n trials. Then **X t**akes the values 0, 1, 2, 3 . . . up to **n**. Therefore, one can view **X** a**s** a discrete random variable. Since number of ways of obtaining **k** successes in **n** trials may be achieved in $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, therefore, binomial probability function may be formulated as $b(n, p, k) = \binom{n}{k} p^k q^{n-k}$.

<u>Illustrative examples:</u>

1. It is known that among the 10 telephone lines available in an office, the chance that any telephone is busy at an instant of time is 0.2. Find the probability that (i) exactly 3 lines are busy, (ii) What is the most probable number of busy lines and compute its probability, and (iii) What is the probability that all the telephones are busy?

Solution:

Here, the experiment about finding the number of busy telephone lines at an instant of time. Let X denotes the number of telephones which are active at a point of time, as there are **n = 10** telephones available; clearly X takes the values right from 0 up to 10. Let **p** denotes the chance of a telephone being busy, then it is given that **p = 0.2**, a finite value. The chance that a telephone line is free is **q = 0.8**. Since a telephone line being free or working is independent of one another, and since this value being same for each and every telephone line, we consider that this problem is based on binomial distribution. Therefore, the required probability mass function is

- $b(10, 0.2, k) = \binom{10}{k} \cdot (0.2)^k \cdot (0.8)^{(10-k)}$ Where k = 0, 1, 2 . . . 10.

(i) To find the chance that 3 lines are busy i.e. P[X = 3] = $\boldsymbol{b}(10, 0.2, 3) = \binom{10}{3} \cdot (0.2)^3 \cdot (0.8)^7$

(ii) With p = 0.2, most probable number of busy lines is $\boldsymbol{n} \cdot \boldsymbol{p} = 10 \cdot 0.2 = 2$. The probability of this number equals $\boldsymbol{b}(10, 0.2, 2) = \binom{10}{2} \cdot (0.2)^2 \cdot (0.8)^8$.

(iii) The chance that all the telephone lines are busy = $(0.2)^{10}$.

2. The chance that a bomb dropped from an airplane will strike a target is 0.4. 6 bombs are dropped from the airplane. Find the probability that (i) exactly 2 bombs strike the target? (ii) At least 1 strikes the target. (iii) None of the bombs hits the target?

<u>Solution:</u> Here, the experiment about finding the number of bombs hitting a target. Let X denotes the number of bombs hitting a target. As **n = 6** bombs are dropped from an airplane, clearly X takes the values right from 0 up to 6.

Let **p** denotes the chance that a bomb hits a target, then it is given that **p = 0.4**, a finite value. The chance that a telephone line is free is **q = 0.6**. Since a bomb dropped from airplane hitting a target or not is an independent event, and the probability of striking a target is same for all the bombs dropped from the plane, therefore one may consider that hat this problem is based on binomial distribution. Therefore, the required probability mass function is $\boldsymbol{b}(10, 0.4, \boldsymbol{k}) = \binom{10}{\boldsymbol{k}} \cdot (0.4)^{\boldsymbol{k}} \cdot (0.8)^{6-\boldsymbol{k}}$.

(i) To find the chance that exactly 2 bombs hits a target, i.e. **P[X = 2]** $= b(10, 0.4, 2) = \binom{10}{2} \cdot (0.4)^2 \cdot (0.8)^4$

(ii) Next to find the chance of the event, namely, at least 1 bomb hitting the target; i.e. $P[X \geq 1] = 1 - P[X < 1] = 1 - P[X = 0] = 1 - (0.6)^6$.

(iii) The chance that none of the bombs are going to hit the target is P[X=0] = $(0.6)^6$.

A discussion on Mean and Variance of Binomial Distribution Function

Let X be a discrete random variable following a binomial distribution function with the probability mass function given by $b(n, p, k) = \binom{n}{k} p^k q^{n-k}$. Consider the expectation of X, namely,

$$E(X) = \sum_{k=0}^{k=n} k \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k \cdot \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} \frac{n(n-1)!}{(k-1)!(n-1+1-k)!} pp^{k-1} q^{(n-1+1-k)}$$

$$= np \cdot \sum_{k=0}^{k=n} \frac{(n-1)!}{(k-1)![(n-1)!-(k-1)!]} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \cdot \sum_{k=1}^{k=n} \frac{(n-1)!}{(k-1)![(n-1)!-(k-1)!]} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \sum_{k=2}^{k=n} \binom{n-1}{k-1} p^{k-1} q^{[(n-1-(k-1)]}$$

$$= np \cdot (p+q)^{n-1}$$

$$= np \text{ as } p+q=1$$

Thus, expected value of binomial distribution function is np.

To find variance of X, consider

$$E\left(X^2\right) = \sum_{k=0}^{k=n} k^2 \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k(k-1+1) \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} k(k-1) \frac{n!}{k!(n-k)!} p^k q^{n-k} + \sum_{k=0}^{k=n} k \cdot \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{k=n} \frac{n(n-1)(n-2)!}{(k-2)![(n-2)!-(k-2)!]} p^2 p^{k-2} q^{[(n-2-(k-2)]} + E(X)$$

$$= n(n-1)p^2 \sum_{k=0}^{k=n} \frac{(n-2)!}{(k-2)![(n-2)!-(k-2)!]} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$$= n(n-1)p^2 \sum_{k=2}^{k=n} \frac{(n-2)!}{(k-2)![(n-2)!-(k-2)!]} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$$= n(n-1)p^2 \sum_{k=2}^{k=n} \binom{n-2}{k-2} p^{k-2} q^{[(n-2-(k-2)]} + np$$

$= n(n-1)p^2 (p+q)^{n-2} + np$. Since **p +q = 1**, it follows that

$$= n(n-1)p^2 + np$$

Therefore, $\mathrm{Var}(X) = E\left(X^2\right) - \{E(X)\}^2$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2 p^2 - np^2 + np - n^2 p^2$$

$= np - np^2 = np(1-p) = npq$. Hence, standard deviation of binomially distributed random variable is $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{npq}$.

A DISCUSSION ON POISSON DISTRIBUTION FUNCTION

This is a limiting case of the binomial distribution function. It is obtained by considering that the number of trials conducted is large and the probability of achieving a success in a single trial is very small i.e. here n is large and p is a small value. Therefore, Poisson distribution may be derived on the assumption that $n \to \infty$ and $p \to 0$. It is found that **Poisson distribution function** is

$$p(\lambda, k) = \frac{e^{-\lambda}\lambda^k}{k!}$$. Here, $\lambda = np$ and $k = 0, 1, 2, 3, \ldots \infty$.

Expectation and Variance of a Poisson distribution function

Consider $E(X) = \sum_{k=0}^{k=\infty} k \cdot p(\lambda, k) = \sum_{k=0}^{k=\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$

$$= \sum_{k=0}^{k=\infty} \frac{e^{-\lambda}\lambda\lambda^{k-1}}{(k-1)!}$$

$= \lambda \cdot e^{-\lambda} \cdot \sum_{k=1}^{k=\infty} \frac{\lambda^{k-1}}{(k-1)!}$. But $\sum_{k=1}^{k=\infty} \frac{\lambda^{k-1}}{(k-1)!} = e^{\lambda}$, therefore it follows that for a Poisson distribution function, $E(X) = \lambda$. Next to find Variance of X, first consider

$$E\left(X^2\right) = \sum_{k=0}^{k=\infty} k^2 \times p(\lambda, k)$$

$$= \sum_{k=0}^{k=\infty} k^2 \times \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1+1) \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} + \sum_{k=0}^{k=\infty} k\,\frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=0}^{k=\infty} k(k-1)\frac{e^{-\lambda}\lambda^k}{k!} + E(X)$$

$$= \sum_{k=0}^{k=\infty} \lambda^2 e^{-\lambda}\frac{\lambda^{k-2}}{(k-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{k=\infty} \frac{\lambda^{k-2}}{(k-2)!} + \lambda$$

$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda$. Thus, $E\left(X^2\right) = \lambda^2 + \lambda$. Hence, Variance of the Poisson distribution function is $\mathrm{Var}(X) = E\left(X^2\right) - \{E(X)\}^2 = \lambda$. The standard deviation is $\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{\lambda}$

<u>Illustrative Examples:</u>

1. **It is known that the chance of an error in the transmission of a message through a communication channel is 0.002. 1000 messages are sent through the channel; find the probability that at least 3 messages will be received incorrectly.**

Solution: Here, the random experiment consists of finding an error in the transmission of a message. It is given that **n = 1000** messages are sent, a very large number, if **p** denote the probability of error in the transmission, we have

p = 0.002, relatively a small number, therefore, this problem may be viewed as Poisson oriented. Thus, average number of messages with an error is $\lambda = np = 2$. Therefore, required probability function is .= $p(2,k) = \dfrac{e^{-2}2^k}{k!}$, k = $0, 1, 2, 3, \ldots \infty$. Here, the problem is about finding the probability of the event, namely,

$$P(X \geq 3) = 1 - P(X < 3) = 1 - \{P[X = 0] + P[X = 1] + P[X = 2]\}$$

$$= 1 - \left[\sum_{k=0}^{k=2} \frac{e^{-2}2^k}{k!}\right]$$

$$= 1 - e^{-2}\{1 + 2 + 2\} = 1 - 5e^{-2}$$

2. **A car hire –firm has two cars which it hires out on a day to day basis. The number of demands for a car is known to be Poisson distributed with mean 1.5. Find the proportion of days on which (i) There is no demand for the car and (ii) The demand is rejected.**

Solution: Here, let us consider that random variable X as the number of persons or demands for a car to be hired. Then X assumes the values 0, 1, 2, 3. … . . . It is given that problem follows a Poisson distribution with mean, $\lambda = 1.5$. Thus, required probability mass function may be written as $p(1.5,k) = \dfrac{e^{-1.5}(1.5)^k}{k!}$.

(i) Solution to I problem consists of finding the probability of the event, namely

P[X = 0] = $e^{-1.5}$ **.**

(ii) The demand for a car will have to be rejected, when 3 or more persons approaches the firm seeking a car on hire. Thus, to find the probability of the event $P[X \geq 3]$. Hence, $P[X \geq 3] = 1 - P\{X < 3] = 1 - P[X = 0,1,2]$ = $e^{-1.5}\left(1 + 1.5 + \frac{(1.5)^2}{2}\right)$.

<u>Illustrative examples based on Continuous Random Variable and it's Probability Density Function</u>

1. **Suppose that the error in the reaction temperature, in ^{o}C, for a controlled laboratory experiment is a R.V. X having the p.d.f**

$$f(x) = \begin{cases} \dfrac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$$

Find (i) F(x) and (ii) use it to evaluate P (0<X≤1).

Solution: Consider $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)dt$

Case (i) x ≤ -1 $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{x} 0dt = 0$

Case (ii) -1< x < 2

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{x} f(t)dt = 0 + \int_{-1}^{x} \frac{t^2}{3} dt = \left.\frac{t^3}{9}\right|_{-1}^{x} = \frac{x^3+1}{9}.$$

Case (iii) x = 2 $F(x) = \int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{-1} f(t)dt + \int_{-1}^{2} f(t)dt + \int_{2}^{x} f(t)dt$

$$= 0 + \int_{-1}^{2} \frac{t^2}{3} dt + 0 = \left.\frac{t^3}{9}\right|_{-1}^{2} = \frac{8+1}{9} = 1$$. **Therefore,**

$$F(x) = \begin{cases} 0, & x \le -1 \\ \dfrac{x^3+1}{9}, & -1 < x < 2 \\ 1, & x \ge 2. \end{cases}$$

2. If the p.d.f of a R.V. X having is given by $f(x) = \begin{cases} 2kxe^{-x^2}, & for\ x > 0 \\ 0, & for\ x \le 0. \end{cases}$

Find (a) the value of *k* and (b) distribution function F(X) for X.

$$\text{WKT} \int_0^\infty 2kxe^{-x^2} dx = 1$$

$$\Rightarrow \int_0^\infty ke^{-t} dt = 1 (\text{put } x^2 = t)$$

$$\Rightarrow \quad ke^{-t}\Big|_0^\infty = 1$$

$$\Rightarrow \quad (0 + k) = 1 \Rightarrow k = 1$$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)dt = 0, \quad \text{if } x \le 0$$

$$= \int_{-\infty}^{0} f(t)dt + \int_0^x f(t)dt, \quad \text{if } x > 0$$

$$= 0 + \int_0^x 2te^{-t^2} dt = (-e^{-z})_0^{x^2} = (1 - e^{-x^2}).$$

$$F(x)=\begin{cases}1-e^{-x^2}, & \text{for } x\ge 0\\ 0, & \text{otherwise.}\end{cases}$$

3. Find the C.D.F of the R.V. whose P.D.F is given by

$$f(x)=\begin{cases}\dfrac{x}{2}, & \text{for } 0<x\le 1\\ \dfrac{1}{2}, & \text{for } 1<x\le 2\\ \dfrac{3-x}{2}, & \text{for } 2<x\le 3\\ 0, & \text{otherwise}\end{cases}$$

Solution: Case (i) $x \le 0$ $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{0} 0dt=0$

Case (ii) $0 < x \le 1$ $F(x)=\int_{-\infty}^{x} f(t)dt=\int_{-\infty}^{0} 0dt+\int_{0}^{x}\frac{t}{2}dt=\frac{x^2}{4}$

Case (iii) $1 < x \le 2$ $F(x)=\int_{-\infty}^{0} f(t)dt+\int_{0}^{1} f(t)dt+\int_{1}^{x} f(t)dt$

$$=0+\int_{0}^{1}\frac{t}{2}dt+\int_{1}^{x}\frac{1}{2}dt=\frac{2x-1}{4}$$

Case (iv) $2 < x \le 3$ $F(x)=\int_{-\infty}^{0} f(t)dt+\int_{0}^{1} f(t)dt+\int_{1}^{2} f(t)dt+\int_{2}^{x} f(t)dt$

$$F(x)=\frac{6x-x^2-5}{4}$$

Case (v) for $x > 3$, $F(x) = 1$. Therefore,

$$F(x)=\begin{cases}0, & \text{if } x\le 0\\ \dfrac{x^2}{4}, & \text{if } 0<x\le 1\\ \dfrac{2x-1}{4}, & \text{if } 1<x\le 2\\ \dfrac{6x-x^2-5}{4}, & \text{if } 2<x\le 3\\ 1, & \text{if } x>3\end{cases}$$

4. The trouble shooting of an I.C. is a R.V. X whose distribution function is given by $F(x)=\begin{cases}0, & \text{for } x\le 3\\ 1-\dfrac{9}{x^2}, & \text{for } x>3.\end{cases}$

If *X* denotes the number of years, find the probability that the I.C. will work properly

(a) less than 8 years
(b) beyond 8 years
(c) anywhere from 5 to 7 years
(d) Anywhere from 2 to 5 years.

Solution: We have $F(x)=\int_0 f(t)dt=\begin{cases}0, & \text{for } x\le 3\\ 1-\dfrac{9}{x^2}, & \text{for } x>3.\end{cases}$

For (a): $P(x\le 8)=\int_0^8 f(t)dt=1-\dfrac{9}{8}=0.8594$

For Case (b): $P(x > 8) = 1 - P(x\le 8) = 0.1406$

For Case (c): $P(5\le x\le 7) = F(7) - F(5) = (1-9/7^2) - (1-9/5^2) = 0.1763$

For Case (d): $P(2\le x\le 5) = F(5) - F(2) = (1-9/5^2) - (0) = 0.64$

5. **A continuous R.V. X has the distribution function is given *by***

$$F(x) = \begin{cases} 0, & x \le 1 \\ c(x-1)^4, & 1 \le x \le 3 \\ 1, & x > 3. \end{cases}$$

Find c and the probability density function.

Solution: We know that $f(x) = \dfrac{d}{dx}[F(x)]$

$$\therefore \quad f(x) = \begin{cases} 0, & x \le 1 \\ 4c(x-1)^3, & 1 \le x \le 3 \\ 0, & x > 3. \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} 4c(x-1)^3, & 1 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

Since we must have $\int_{-\infty}^{\infty} f(x)dx = 1,$

$$\int_1^3 4c(x-1)^3 dx = 1 \quad \Rightarrow \left[c(x-1)^4 \right]_1^3 = 1$$

$$\Rightarrow 16c = 1 \quad \therefore c = \frac{1}{16}$$

Using this, one can give the probability function just by substituting the value of c above.

A discussion on some standard distribution functions of continuously distributed random variable:

This distribution, sometimes called the negative exponential distribution, occurs in applications such as reliability theory and queuing theory. Reasons for its use include its memory less (Markov) property (and resulting analytical tractability) and its relation to the (discrete) Poisson distribution. Thus, the following random variables may be modeled as exponential:

- Time between two successive job arrivals to a computing center (often called inter-arrival time)
- Service time at a server in a queuing network; the server could be a resource such as CPU, I/O device, or a communication channel
- Time to failure of a component i.e. life time of a component
- Time required repairing a component that has malfunctioned.

The exponential distribution function is given by, $f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0, \\ 0, & \text{otherwise.} \end{cases}$

The probability distribution function may be written as $F(x) = \int_{-\infty}^{x} f(x)dx$ which may be computed as $F(x) = \begin{cases} 1 - e^{-\lambda x}, & \text{if } 0 < x < \infty \\ 0, & \text{otherwise.} \end{cases}$.

Mean and Variance of Exponential distribution function

$$\textbf{Consider mean } (\mu) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda\left[x\cdot\left(\frac{e^{-\lambda x}}{-\lambda}\right)-1\cdot\left(\frac{e^{-\lambda x}}{-\lambda^{2}}\right)\right]_{0}^{\infty} = -\lambda\left[0-\frac{1}{\lambda^{2}}\right] = \frac{1}{\lambda}$$

$$\text{Consider } E\left(X^{2}\right) = \int_{-\infty}^{\infty} x^{2}\cdot f(x)\,dx = \int_{0}^{\infty} x^{2}\cdot\lambda e^{-\lambda x}\,dx$$

$$= \lambda\left[x^{2}\left(\frac{e^{-\lambda x}}{-\lambda}\right)-2x\left(\frac{e^{-\lambda x}}{-\lambda^{2}}\right)+2\left(\frac{e^{-\lambda x}}{-\lambda^{3}}\right)\right]_{0}^{\infty} = \frac{2}{\lambda^{2}}$$

$$\mathrm{Var}(X) = E\left(X^{2}\right) - \left\{E(X)\right\}^{2} = \frac{1}{\lambda^{2}}.$$

The standard deviation is $\sigma = \sqrt{Var(X)} = \dfrac{1}{\lambda}$.

<u>Illustrative examples based on Exponential distribution function</u>

1. The duration of telephone conservation has been found to have an exponential distribution with mean 2 minutes. Find the probabilities that the conservation may last (i) more than 3 minutes, (ii) less than 4 minutes and (iii) between 3 and 5 minutes.

Solution: Let X denotes the random variable equals number of minute's conversation may last. It is given that X is exponentially distributed with mean 3 minutes. Since for an exponential distribution function, mean is known to be $\frac{1}{\lambda}$, so $\frac{1}{\lambda} = 2$ or $\lambda = 0.5$. The Probability density function can now be written as

$$f(x) = \begin{cases} 0.5e^{-0.5x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(i) To find the probability of the event, namely,

$$P[X > 3] = 1 - P[X \le 3] = 1 - \int_0^3 0.5e^{-0.5x}\,dx$$

(ii) To find the probability of the event, namely $P[X < 4] = \int_0^4 0.5e^{-0.5x}\,dx$.

(iii) To find the probability of the event $P[3 < X < 5] = \int_3^5 0.5e^{-0.5x}\,dx$.

2. in a town, the duration of a rain is exponentially distributed with mean equal to 5 minutes. What is the probability that (i) the rain will last not more than 10 minutes (ii) between 4 and 7 minutes and (iii) between 5 and 8 minutes?

Solution: An identical problem to the previous one. Thus, may be solved on similar lines.

Discussion on Gaussian or Normal Distribution Function

Among all the distribution of a continuous random variable, the most popular and widely used one is normal distribution function. Most of the work in correlation and regression analysis, testing of hypothesis, has been done based on the assumption that problem follows a normal distribution function or just everything normal. Also, this distribution is extremely important in statistical applications because of the central limit theorem, which states that under very general assumptions, the mean of a sample of n mutually

Independent random variables (having finite mean and variance) are normally distributed in the limit $n \to \infty$. It has been observed that errors of measurement often possess this distribution. Experience also shows that during the wear – out phase, component life time follows a normal distribution. The purpose of today's lecture is to have a detailed discussion on the same.

The normal density function has well known bell shaped curve which will be shown on the board and it may be given as $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$

where $-\infty < \mu < \infty$ and $\sigma > 0$. It will be shown that μ and σ are respectively denotes mean and variance of the normal distribution. As the probability or cumulative distribution function, namely, $F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$ has no closed form, evaluation of integral in an interval is difficult. Therefore, results relating to probabilities are computed numerically and recorded in special table called normal distribution table. However, It pertain to the standard normal distribution function by choosing μ and σ and their entries are values of the function, $F_z(z) = \dfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt.$ Since the standard normal distribution is symmetric, it can be shown that

$$F_z(-z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} f(t)\, dt = 1 - F_Z(z) .$$

Thus, tabulations are done for positive values of z only. From this it is clear that

- $P(a \le X \le b) = F(b) - F(a)$
- $P(a < X < b) = F(b) - F(a)$
- $P(a < X) = 1 - P(X \le a) = 1 - F(a)$

Note: Let X be a normally distributed random variable taking a particular value, x, the corresponding value of the standardized variable is given by $z = \frac{x-\mu}{\sigma}$. Hence,

$$F(x) = P(X \le x) = F_z\left(\frac{x-\mu}{\sigma}\right).$$

Illustrative Examples based on Normal Distribution function:

1. In a test on 2000 electric bulbs, it was found that the life of a particular make was normally distributed with an average life of 2040 hours and standard deviation of 60 hours. Estimate the number of bulbs likely to burn for (a) more than 2150 hours, (b) less than 1940 hours and (c) more than 1920 hours and but less than 2060 hours.

Solution:

Here, the experiment consists of finding the life of electric bulbs of a particular make (measured in hours) from a lot of 2000 bulbs. Let X denotes the random variable equals the life of an electric bulb measured in hours. It is given that X follows normal distribution with mean $\mu = 2040$ hours and $\sigma = 60$ hours.

First to calculate $P(X > 2150 \text{ hours}) = 1 - P(X \le 2150)$

$$=1-F_Z(1.8333)=1-0.9664=0.0336$$

Therefore, number of electrical bulbs with life expectancy more than 2150 hours is $0.0336\times 2000\approx 67$.

Next to compute the probability of the event; $P(X<1950 \text{ hours})=F_Z\left(\frac{1950-2040}{60}\right)$

$$=F_Z(-1.5)=1-F_Z(1.5)=1-0.9332=0.0668$$

Therefore, in a lot of 2000 bulbs, number of bulbs with life expectancy less than 1950 hours is 0.0668 * 2000 = 134 bulbs.

Finally, to find the probability of the event, namely,

$$P(1920<X<2060)=F(2060)-F(1920)$$

$$=F_Z\left(\frac{2060-2040}{60}\right)-F_Z\left(\frac{1920-2040}{60}\right)$$

$$=F_Z(0.3333)-F_Z(-2)$$

$$=F_Z(0.3333)-1+F_Z(2)$$

$$=0.6293-1+0.9774=0.6065.$$

Therefore, number of bulbs having life anywhere in between 1920 hours and 2060 hours is 0.6065 * 2000 = 1213.

2. Assume that the reduction of a person's oxygen consumption during a period of Transcendenta Meditation (T.M.) is a continuous random variable X normally distributed with mean 37.6 cc/min and S.D. 4.6 cc/min. Determine the probability that during a period of T.M. a person's oxygen consumption

will be reduced by (a) at least 44.5 cc/min (b) at most 35.0 cc/min and (c) anywhere from 30.0 to 40.0 cc/min.

Solution: Here, X a random variable is given to be following normal distribution function with mean.P $\mu = 37.6$ and $\sigma = 4.6$. Let us consider that X as the random equals the rejection of oxygen consumption during T M period and measured in cc/min.

(i) To find the probability of the event $P[X \geq 44.5] = 1 - F(44.5)$

$$= 1 - F_z\left(\frac{44.5 - 37.6}{4.6}\right)$$

$$= 1 - F_z(1.5)$$

$$= 1 - 0.9332 = 0.0668.$$

(ii) To find the probability of the event, $P[X \leq 35.0] = F(33.5)$

$$= F_z\left(\frac{35.0 - 37.6}{4.6}\right)$$

$$= F_z(-0.5652)$$

$$= 1 - F_z(0.5652)$$

$$= 1 - 0.7123 = 0.2877.$$

(iii) Consider the probability of the event $P[30.0 < X < 40.0]$

$$= F(40) - F(30)$$

$$= F_z\left(\frac{40-37.6}{4.6}\right) - F_z\left(\frac{30-37.6}{4.6}\right)$$

$$= F_z(0.5217) - F_z(-1.6522)$$

$$= 0.6985 - 1 + 0.9505 = 0.6490$$

3. An analog signal received at a detector (measured in micro volts) may be modeled as a Gaussian random variable N (200, 256) at a fixed point in time. What is the probability that the signal will exceed 240 micro volts? What is the probability that the signal is larger than 240 micro volts, given that it is larger than 210 micro volts?

Solution: Let X be a CRV denotes the signal as detected by a detector in terms of micro volts. Given that X is normally distributed with mean 200 micro volts and variance 256 micro volts. To find the probability of the events, namely, (i) P (X > 240 micro volts] and (ii) P[X > 240 micro volts | X > 210 micro volts].

Consider $P[X > 240] = 1 - P[X \le 240]$

$$= 1 - F(240)$$

$$= 1 - F_z\left(\frac{240-200}{16}\right)$$

$$= 1 - F_z(2.5)$$

$= 1 - 0.9938$

$= 0.00621$

Next consider P**[X > 240 | X > 210]**

$$= \frac{P[X > 240 \text{ and } X > 210]}{P[X > 210]}$$

$$= \frac{P[X > 240]}{P[X > 210]} = \frac{1 - P[X \le 240]}{1 - P[X \le 210]}$$

$$= \frac{1 - F_z\left[\frac{240-200}{16}\right]}{1 - F_z\left[\frac{210-200}{16}\right]} = \frac{1 - F_z(2.5)}{1 - F_z(0.625)}$$

$$= \frac{1 - 0.9939}{1 - 0.73401} = 0.2335$$