MODULE - I

COMPLEX VARIABLES

Complex number:

- The Real and Imaginary part of a complex number $z = x + iy$ are x and y respectively, and we write

$$\operatorname{Re} z = x \text{ and } \operatorname{Im} z = y$$

- We may represent the complex number z in polar form:

$$z = r[\cos\theta + i\sin\theta]$$

- Where $x = r\cos\theta$, $y = r\sin\theta$,r is called the absolute value and θ is the argument of Z.

Now

$$z = r\, e^{i\theta}$$

$$|z| = r\left|e^{i\theta}\right|$$

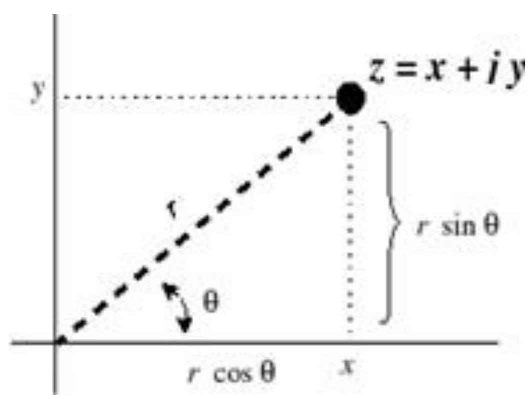
$$|z| = r \quad and \quad \arg z = \theta$$

- Geometrically $|z|$ is the distance of the point z from the origin. For any complex number

$$z = x + i\,y$$

$$|z| = \sqrt{x^2 + y^2}$$

$$r = \sqrt{x^2 + y^2}$$



- Distance between two points, $z_1 = x_1 + iy_1 \quad and \quad z_2 = x_2 + iy_2$

Now $z_1 - z_2 = (x_1 - x_2) + i\,(y_2 - y_1)$ is a complex number.

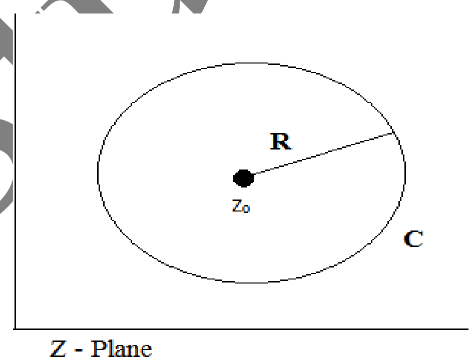
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- **Equations and inequalities of curves and regions in the complex plane:**
- Consider $|z - z_0| = R$ ---(1)
- Where $z = x+iy$ is any point and $z_0 = x_0 + iy_0$ is a fixed point, R is a given real constant.

$|z - z_0| = R$ **OR** $z - z_0 = R\,e^{i\theta} \quad 0 \le \theta \le 2\pi$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = R$$

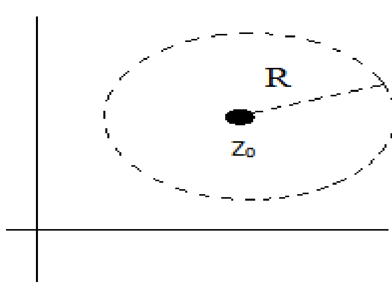
$$(x - x_0)^2 + (y - y_0)^2 = R^2 \text{ ------ } (2)$$



Equation (2) represents a circle C of radius R with the center at a point (x_0, y_0). Hence equation (1) represents a circle C center at z_0 with radius R in the complex plane

Consequently we have,

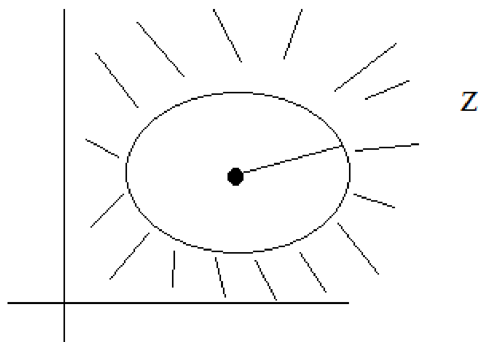
1. The inequality $|z - z_0| < R$, holds for any point z inside C; ie. $|z - z_0| < R$ represents set of complex points lies inside C or interior points of C. such a region is called a circular disk or more precisely open circular disk or open set.



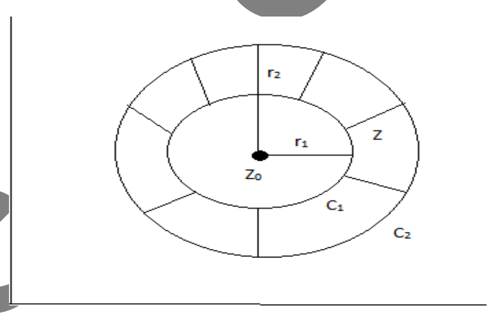
Note: If R is very small say δ>0 (no matter, how small but not zero) then $|z-z_0|<\delta$ is called a nhd of the point z_0.

2. The inequality $|z-z_0|\le R$, holds for any z inside and on the C. such a region is called circular disk or closed set [$|z-z_0|\le R$ consists interior of C and C itself].

3. The inequality $|z-z_0|>R$ represents exterior of the circle C.



4. The inequality $r_1<|z-z_0|<r_2$ represents a region between two concentric circles C_1 and C_2 of radii r_1 and r_2 respectively. Where z_0 is the center of circles. Such a region is called an open circular ring or annular region.



5. Suppose z_0 = 0, then $|z|=R$ represents a circle C of radius R with center at the origin in the complex plane.

Consequently we have the following:

The equation $|z|=1$ represents the unit circle of radius 1 with center at the origin.

a) $|z|<1$: represents the open unit disk.

b) $|z|\leq 1$ **:** represents the closed unit disk.

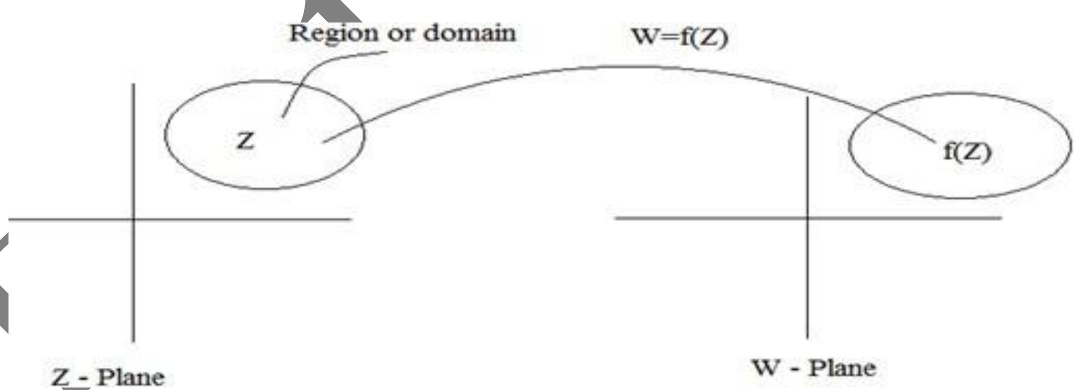
[Students become completely familiar with representations of curves and regions in the complex plane]

Complex variable:

- If x and y are real variables, then $z=x+iy$ is said to be a complex variable.

Complex Function:

- If, to each value of a complex variable z in some region of the complex plane or z-plane there corresponds one or more values of W in a well defined manner, then W is a function of z defined in that region (domain), and we write $W=f(z)$.



Observation:

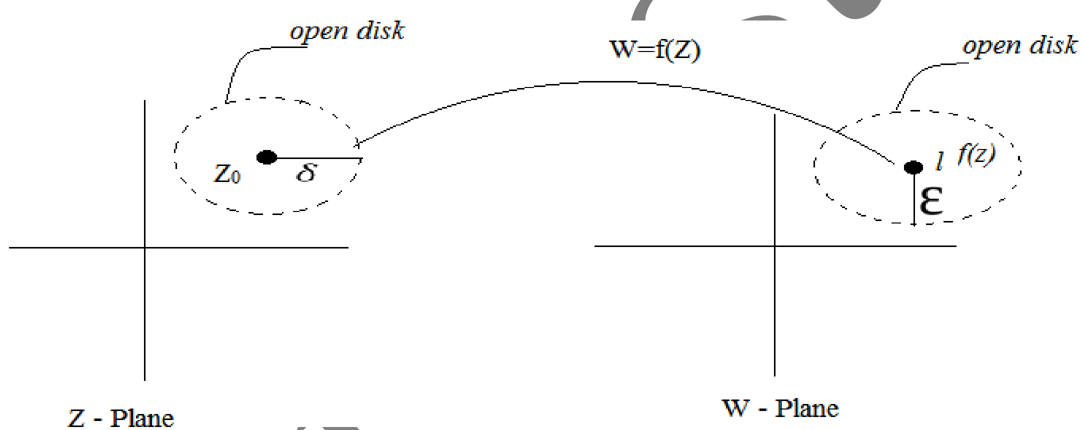
- The real and imaginary part of a complex function $W = f(z) = u + iv$ are u and v which are depends on:

  i. x,y in Cartesian form.

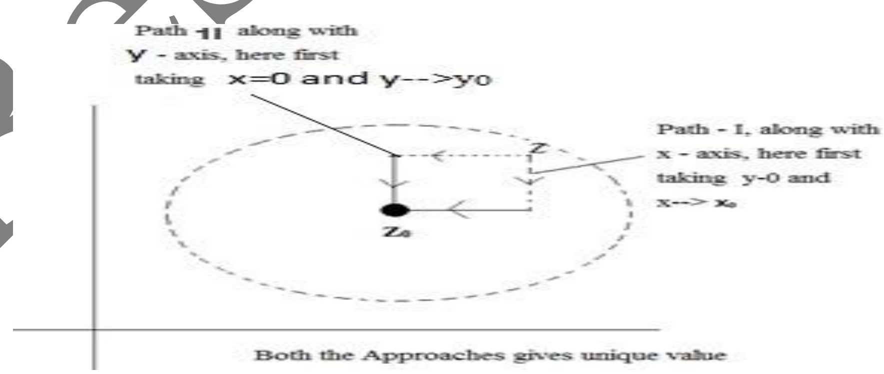
  ii. r, θ in polar form.

Limit:

A complex valued function *f(z)* is said to have the limit l as z approaches to z_0 (except perhaps at z_0) and if every positive real number ∈>0 (no matter, how small but not zero) we can find a positive real number δ>0 such that $|f(z) - l| < \varepsilon$ whenever $|z - z_0| < \delta$ for all values $z \neq z_0$ or $\lim_{Z \to Z_0} f(z) = f(z_0)$



- $z \to z_0$ means that, **z** approaches to z_0 through independent of path.



$z = z_0$

Continuity of : A complex function $W = f(z)$ is said to be continuous at a point z_o if

i) $f(z_0)$ is exists.

ii) $\lim\limits_{Z \to Z_0} f(z) = f(z_0)$

Note: If $f(z)$ is said to be continuous in any region R of the z-plane, if it is continuous at every point of that region.

Derivative of f(z):

A complex function f(z) is said to be differentiable at $z=z_0$ if $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists and is unique. This limit is then called the derivative of f(z) at $z=z_0$ and denoted by $f'(z_0)=\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ or $f'(z_0)=\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ where $\delta z = z - z_0$.

Theorem: The necessary conditions for the derivative of the function w = f(z) to exist for all values of z in a region R,

i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous function of x and y in R.

ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. the relation (ii) are known as Cauchy- Riemann equations or briefly C-R Equations.

Proof: If f(z) possesses a unique derivative at any point z in R, then

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

In Cartesian form $f(z) = u(x, y) + i\, v(x, y)$

$\delta z = \delta x + i\,\delta y$, *and*

$f(z+\delta z) = u(x+\delta x, y+\delta y) + i\, v(x+\delta x, y+\delta y)$

$$f'(z) = \lim_{\delta z \to 0}\left\{\frac{[u(x+\delta x, y+\delta y) + i\, v(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta x + i\delta y}\right\}$$

$$f'(z) = \lim_{\delta z \to 0}\left\{\frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x + i\delta y} + i\,\frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x + i\delta y}\right\} --- (1)$$

Let us consider the limit $\delta z \to 0$ along the path parallel to the x-axis (for which $\delta y = 0$), then

RHS of (1) becomes $$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \frac{v(x+\delta x, y) - v(x,y)}{\delta x} \right\}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} --- (2)$$

Let us consider the limit $\delta z \to 0$ along the path parallel to the y-axis (for which $\delta x = 0$), then RHS of (1)

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \frac{v(x, y+\delta y) - v(x,y)}{i\delta y} \right\}$$

$$f'(z) = \lim_{\delta z \to 0} \left\{ \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + \frac{v(x, y+\delta y) - v(x,y)}{\delta y} \right\}$$

$$f'(z) = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} --- (3)$$

Now existence of $f'(z)$ requires equality of (2) and (3)

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary part from both the sides.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}.$$

Analytic function:

A complex function *f(z)* is said to be analytic at a point $z = z_0$ if it is differentiable at z_0 as well as in a nhd of the point z_0. An analytic function is also called a regular function or an holomorphic function.

Theorem (2): If $f(z)=u+iv$ is analytic at a point $z=x+iy$, then u and v satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$ at that point.

Proof:

f(z) is analytic means that *f(z)* possesses a unique derivative at a point $z=x+iy$. (proof of theorem(1) follows)

Cauchy-Riemann equations in Polar form:

Property: show that the polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Solution:

Complex variable z in polar form is

$$z=r\,e^{i\theta}---(1)$$

W=f(z)

$$u+i\,v=f(re^{i\theta})----(2)$$

where u and v are functions of r θ

$$\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}=f'(re^{i\theta}).\quad e^{i\theta}---(3)$$

$$\frac{\partial u}{\partial \theta}+i\,\frac{\partial v}{\partial \theta}=f'(re^{i\theta}).\quad rie^{i\theta}---(4)$$

$$\frac{\partial u}{\partial \theta}+i\,\frac{\partial v}{\partial \theta}=ri\left[\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}\right]$$

$$\frac{\partial u}{\partial \theta}+i\,\frac{\partial v}{\partial \theta}=\left[ri\frac{\partial u}{\partial r}-r\frac{\partial v}{\partial r}\right]$$

Equating real and imaginary parts we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \quad and \quad \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

Note-1: The necessary conditions for *f(z)* to be analytic are $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ *and* $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

these two relations are called Cauchy-Riemann Equations.

Note-2: The sufficient conditions for *f(z)* to be analytic are, four partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ must exist and must be continuous at all points of the region.

Example-1:

- Show that *f(z)=Re z* is not analytic.

Solution: *f(z) = Re z =x*

u=x and v=0

$$\frac{\partial u}{\partial x}=1,\quad \frac{\partial v}{\partial x}=0,\quad \frac{\partial u}{\partial y}=0,\quad \frac{\partial v}{\partial x}$$

C-R equation $\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y}$ $\frac{\partial u}{\partial y}\neq-\frac{\partial v}{\partial x}$, are not satisfied

Hence *f(z)=Re z=x* is not analytic similarly *f(z)=Im z=y* is not analytic

Property-1: The real and imaginary parts of an analytic functions *f(z)=u+iv* in some region of the z-plane are solutions of Laplace's equations in two variables.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad and \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Solution: *f(z)=u+iv* is an analytic function, then

$$(By\ C-R\ Equation)\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}----(1)$$

Consider $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}----(2),\qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}----(3)$

Diff *(2)* with respect to *x*

Diff *(3)* with respect to *y*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} ----(4)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} ----(5)$$

Adding *(4)* and *(5)*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 ----(6)$$

Diff *(2)* with respect to y $\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial y \partial x} ----(7)$

Diff *(2)* with respect to *x* $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} ----(8)$

Adding *(7)* and *(8)* we get $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 ----(9)$

- Thus both functions *u(x,y)* and *v(x,y)* satisfy the Laplace's equations in two variables. For this reasons, they are known as Harmonic functions or Conjugate Harmonic function.

Polar form: If $f(z)=u(r, \theta)+i\, v(r, \theta)$ is an analytic function, then show that *u* and *v* satisfy Laplace's equation in polar form.

- Laplace equation in Polar form in two variables,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

We have C-R equation in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} ----(1)$$

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} ----(2)$$

Differentiate *(1)* with respect to *r*, $\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} ----(3)$

Differentiate *(2)* with respect to *θ*, $\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} ----(4)$

using *(4)* and *(1)* on *RHS* of Equation *(3)*, we get

$$\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\left(\frac{\partial u}{\partial r}\right) + \frac{1}{r}\left(-\frac{1}{r}\frac{\partial^2 u}{\partial \theta}\right)$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

➢ *Hence* ***u*** is Harmonic

From *(1)* we get, $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial \theta}$

Differentiate with respect to θ $\frac{\partial^2 v}{\partial \theta^2} = r\frac{\partial^2 u}{\partial \theta \partial r} ----(5)$

From *(2)* we get $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} ----(6)$

Differentiate with respect to r $\frac{\partial^2 v}{\partial r^2} = +\frac{1}{r^2}\frac{\partial u}{\partial \theta} - \frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} ----(7)$

using *(5)*,*(6)* on RHS of *(7)*

$$\frac{\partial^2 v}{\partial r^2} = \frac{1}{r}\left(-\frac{\partial v}{\partial r}\right) - \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 v}{\partial \theta^2}\right) = 0$$

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$$

Hence **v** is Harmonic

Orthogonal System:

- Two curves are said to be orthogonal to each other when they intersect at right angles at each of their point of intersections.

Property: If *w=f(z)=u+iv* be an analytic function then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system.

Solution: *f(z)=u+iv* is an analytic functions.

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}\right\} ----C-\text{R equation}$$

$$u(x, y) = c_1$$

differentiate with respect to x, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = m_1 ----(2)$$

differentiate w.r.t, x we get

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2 ---(3)$$

$$\therefore \quad m_1.m_2 = \frac{+\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}} \times \frac{+\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}}$$

$$= \frac{\dfrac{\partial v}{\partial y}}{-\dfrac{\partial v}{\partial x}} \times \frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial y}} \quad \text{(By C-R Equations)}$$

$m_1.m_2 = -1$, form an orthogonal system

Polar form: Consider $u(r,\theta) = c_1 ---(1)$ and $v(r,\theta) = c_2 ---(2)$

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} &= -r \frac{\partial v}{\partial r} \end{aligned}\right\} ----(3) \text{ C-R Equations}$$

differentiate (1) w.r.t. θ

$$\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial r} \frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\dfrac{\partial u}{\partial \theta}}{\dfrac{\partial u}{\partial r}} ------(4)$$

$\tan\phi_1 = \dfrac{r}{\dfrac{dr}{d\theta}}$ where ϕ_1 being the angle between the radius vector and the tangent to the curve(1)

$$\tan\phi_1 = \frac{r}{\frac{-\frac{\partial u}{\partial \theta}}{\frac{\partial u}{\partial r}}}$$

$$\tan\phi_1 = -\frac{r\frac{\partial u}{\partial r}}{\frac{\partial u}{\partial \theta}} \quad -----(5)$$

Differentiate (2) w. r. t. θ

$$\frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r}\frac{dr}{d\theta} = 0$$

$$\frac{dr}{d\theta} = \frac{-\frac{\partial v}{\partial \theta}}{\frac{\partial v}{\partial r}}$$

$\tan\phi_2 = \frac{r}{\frac{dr}{d\theta}}$, where ϕ_2 being the angle between the radius and the tangent to the curve(2)

$$\tan\phi_1 \times \tan\phi_2 = \frac{r\dfrac{\partial u}{\partial r}}{\dfrac{\partial u}{\partial \theta}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$$= \frac{r.\dfrac{1}{r}\dfrac{\partial v}{\partial \theta}}{-r\dfrac{\partial \theta}{\partial r}} \times \frac{r\dfrac{\partial v}{\partial r}}{\dfrac{\partial v}{\partial \theta}}$$

$= -1$ form an orthogonal system

Note: We have $z = x + iy \quad and \quad \overline{z} = x - iy$

$$Now \quad x = \frac{1}{2}(z + \overline{z})$$

$$y = \frac{1}{2i}(z - \overline{z})$$

$Consider \quad f(z) = u(x, y) + i \quad v(x, y) ----(1)$

$$f(z) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + i \quad v\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

$put \quad z = \overline{z}$ we get

$f(z) = u(z, 0) + i \quad v(z, 0) ----(2)$

$\therefore$(2) is same as (1) if we replace x by z and y by 0

Similarly in polar form if we replace r by z and θ by 0 in $f(z) = u(r,\theta) + i \quad v(r,\theta)$

This is due to Milne-Thomson

$$Note: \ (i) \ \sin(i\,x) = i \sin h\, x \quad or \quad \sin hx = \frac{1}{i}[\sin(i\,x)]$$

$$(ii) \cos(i\,x) = \cos hx$$

Example:1

Show that *f(z)=sin z* is analytic and hence find, $f'(z)$

Solution: *f(z) = sin(z)*

=sin(x+iy)

=sin(x)cos(iy)+cos(x)sin(iy)

f(z)=sin x cos hy + i cos x sin hy

Equating *real* and *imaginary* parts *u=sinx coshy* and *v=cosx sin hy--(1)*

u and *v* satisfies necessary conditions $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cos x \cos h\, y + i(-\sin x)\sin h\, y) ----(*)$$

$$= \cos(x)\cos(iy) - i\sin x.\frac{1}{i}\sin(iy)$$

$$= \cos(x)\cos(iy) - \sin x \sin(iy)$$

$$= \cos(x+iy)$$

$$f'(z) = \cos(z) \qquad \therefore \frac{d[\sin z]}{dz} = \cos z$$

or By Milne's Thomson method replace x by z and y by 0 in (*)

$$f'(z) = \cos(z).1 - 0 \qquad \therefore f'(z) = \cos(z) \quad or \quad \frac{d[\sin z]}{dz} = \cos z$$

2) *Show that* $w = z + e^z$ is analytic, hence find $\dfrac{dw}{dz}$

Solution: Let $w = f(z) = u + iv$.

$w = (x + e^x \cos y) + i(y + e^x \sin y)$

Equating real and imaginary parts

$u = (x + e^x \cos y),\ v = (y + e^x \sin y)$

u and v satisfies C-R equations

consider

$$\frac{dw}{dz} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$
$$= (1 + e^x \cos y) + i(e^x \sin y)$$
$$= 1 + e^x[\cos y + i \sin y] - - - (1)$$
$$= 1 + e^x . e^{iy}$$
$$= 1 + e^z$$
$$\frac{d[z + e^z]}{dz} = 1 + e^z$$

Or By Milne's-Thomson method replace x by z and y by 0 in (1), we get derivative of $z + e^z$

Example-3:

show that $w = \log(z)$ *is* analytic, hence find $f'(z)$

$w = \log[re^{i\theta}]$

$w = \log(r) + i\theta$ equating real and imaginary parts

$u = \log(r)$ and $v = \theta$, u and v satisfies C-R equation in polar form.

consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$
$$= e^{-i\theta}\left[\frac{1}{r}\right]$$
$$= \frac{e^{-i\theta}}{r}$$
$$f'(z) = \frac{1}{re^{i\theta}} - - - - (1)$$
$$f'(z) = \frac{1}{z}$$
$$\therefore \frac{d[\log z]}{dz} = \frac{1}{z}$$

or by Milne's Thomson method replace r by z and θ by 0 in RHS of (1), we get $\dfrac{d[\log z]}{dz} = \dfrac{1}{z}$

Cauchy's-Riemann equations in Cartesian form

Statement: The real and imaginary part of an analytic function *f(z)=u(x,y)+iv(x,y)* satisfies Cauchy's-Riemann equations.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at each point

Note: A function f(z) is analytic, then

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ limit approaches along the x-axis

and $f'(z) = \frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y}$ limit approaches along the y-axis

Example: The function $f(z) = z^2$ is analytic for all z, and $f'(z) = 2z$

Solution:

$f(z) = (x^2 - y^2) + i\,2xy$ is analytic every in the complex plane.

$u = x^2 - y^2$ and $v = 2xy$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= 2x + i2y \\ &= 2(x+iy) \\ &= 2z \end{aligned}$$

$$\therefore \frac{d(z^2)}{dz} = 2z$$

Note: If *f(z)=u(r,θ)+iv(r,θ)* then Cauchy-Riemann equation in polar form:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

where $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$ limit approaches along the radial line and

$$f'(z) = \frac{e^{-i\theta}}{r}\left[\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right]$$ a limit approach along angular path.

Construction of Analytic Function:

Construction of analytic function $f(z) = u + iv$ when u or v or $u \pm v$ is given.

Example1: Find the Analytic Function *f(z),* whose real part is $e^{2x}[x\cos 2y - y\sin 2y]$.

Solution:

Given $u = e^{2x}[x\cos 2y - y\sin 2y]----(1)$

Differentiate (1) w.r.t. x

$$\frac{\partial u}{\partial x} = e^{2x}\left[\cos 2y\right] + 2e^{2x}\left[x\cos 2y - y\sin 2y\right]----(2)$$

Differentiate (1) w.r.t. y

$$\frac{\partial u}{\partial y} = e^{2x}\left[-2.x.\sin 2y - y.2\cos 2y - \sin 2y\right]----(3)$$

$$Consider \quad f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} ----(4)$$

By C-R Equations replace $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$$f'(z) = \frac{\partial u}{\partial x} - i\,\frac{\partial u}{\partial y} ----(5)$$

using (2) and (3) on RHS (5)

$$f'(z) = e^{2x}\left[\cos 2y + 2x\cos 2y - 2y\sin 2y\right] + i\, e^{2x}\left[2x\sin 2y + 2y\cos 2y + \sin 2y\right]$$

By Milne's Method replace x by z and y by 0

$$f'(z) = e^{2z}[1 + 2z]$$

$$f'(z) = e^{2z} + 2e^{2z}.z$$

int*egrate* we get

$$f(z) = \frac{1}{2}e^{2z} + 2\left[\frac{e^{2z}}{2}.z - \frac{e^{2z}}{4}\right] + c$$

$$f(z) = \frac{1}{2}e^{2z} + ze^{2z} - \frac{1}{2}e^{2z} + c$$

$$f(z) = ze^{2z} + c$$

Note: $u + iv = (x + iy)e^{2x}.e^{i2y} + c$

$= e^{2x}(x + iy)(\cos 2y + i\sin 2y)$

$u + iv = e^{2x}\left[(x\cos 2y - y\sin 2y) + i(y\cos 2y + x\sin 2y)\right] + c$

$\therefore u = e^{2x}[x\cos 2y - y\sin 2y] + c$

$v = e^{2x}(y\cos 2y + x\sin 2y)$

Taking $c = 0$ *we get*

$u = e^{2x}[x\cos 2y - y\sin 2y]$ which is real part

and $v = e^{2x}[y\cos 2y + x\sin 2y]$ is imaginary part of a required analytic function $f(z)$

2) Find the Analytic function whose real part is $\dfrac{\sin 2x}{\cos 2y - \cos 2x}$

$$Solution: u = \frac{\sin 2x}{\cosh 2y - \cos 2x} ------(1)$$

Differentiate w.r.t. x

$$\frac{\partial u}{\partial x} = \frac{(\cosh 2y - \cos 2x).2\cos 2x - \sin 2x[+2\sin 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cosh 2y \cos 2x - 2[\cos^2(2x) + \sin^2 2x]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial x} = \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2x)^2} ------(2)$$

Differentiate (1) w.r.t. y

$$\frac{\partial u}{\partial y} = \frac{\sin 2x[-(2\sinh 2y)]}{(\cosh 2y - \cos 2x)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} ------(3)$$

Consider $f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x}$

By C-R equation replace $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$ $\qquad f'(z) = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$

$$f'(z)=\frac{[2\cos 2x\cosh 2y-2]+i2\sin 2x\sinh 2y}{(\cosh 2y-\cos 2x)^2}$$

By Milne's Thomson method replace x by z and y by 0

$$f'(z)=\frac{2[\cos 2z-1]+i.0}{(1-\cos 2z)^2}$$

$$f'(z)=\frac{-2[1-\cos 2z]}{(1-\cos 2z)^2}$$

$$f'(z)=\frac{-2}{[1-\cos 2z]}$$

$$f'(z)=\frac{-2}{2\sin^2 z}$$

$$f'(z)=-\operatorname{cosec}^2 z$$

integrate

$$f(z)=+\cot z+c$$

3) Construct the analytic function whose imaginary part is $\left(r-\frac{1}{r}\right)\sin\theta,\ r\neq 0.$

Hence find the Real part.

Solution: Given $v=\left(r-\frac{1}{r}\right)\sin\theta$ ----(1)

Differentiate (1) *w.r.t.* θ

$$\frac{\partial u}{\partial \theta}=\left(r-\frac{1}{r}\right)\cos\theta \text{ ------------(2)}$$

Differentiate (1) *w.r.t.* r

$$\frac{\partial u}{\partial r}=\left(1+\frac{1}{r^2}\right)\sin\theta \text{ ------------(3)}$$

Consider $f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right]$ ----(4)

By C-R Equation replace $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ *on*

RHS of (4) *we get*

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\frac{\partial v}{\partial \theta} + i\,\frac{\partial v}{\partial r}\right]$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta + i\left(1 + \frac{1}{r^2}\right)\sin\theta\right]$$

By Milne's method replace r by z and θ by 0

$$f'(z) = e^{0}\left[\frac{1}{z}\left(z - \frac{1}{z}\right).1 + i.0\right]$$

$$f'(z) = \left(1 - \frac{1}{z^2}\right)$$

Integrate we get

$$f(z) = z + \frac{1}{z} + ic$$

To find real part: Consider $f(z) = re^{i\theta} + \frac{1}{re^{i\theta}} + ic$

$$u + iv = (r\cos\theta + ir\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) + ic$$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left[\left(r - \frac{1}{r}\right)\sin\theta + c\right]$$

Equating real and imaginary parts

$$u = \left(r + \frac{1}{r} \right) \cos\theta$$

$$v = \left(r - \frac{1}{r} \right) \sin\theta + c$$ to get actual imaginary part of an analytical function

$f(z) = u + iv$ *taking* $c = 0$

$$\therefore v = \left(r - \frac{1}{r} \right) \sin\theta$$

4) Find an analytic function $f(z)$ as a function of z

given that the sum of real and imaginary part is $x^3 - y^3 + 3xy(x - y)$

Solution: The sum of real and imaginary part is given by

$$u + v = x^3 - y^3 + 3xy(x - y) -------(1)$$

Differentiate (1) *w.r.t.* x

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 - 0 + 3xy + 3y(x - y)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x - y) ----(2)$$

Differentiate (1) *w.r.t.* y

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 - 3y^2 + 3xy(-1) + 3x(x - y)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -3y^2 - 3xy + 3x(x - y) ------(3)$$

By C-R Equation replace $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad in(3)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y) ------(4)$$

Consider

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + (x-y)3(x+y)$$

$$2\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 3x^2 - 3y^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 ----------(5)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 3x^2 + 3xy + 3y(x-y)$$

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = -3y^2 - 3xy + 3x(x-y)$$

$$2\frac{\partial v}{\partial x} = 3x^2 + 3y^2 + 6xy + (x-y).3(y-x)$$

$$= 3x^2 + 3y^2 + 6xy - 3(x-y)^2$$

$$= 3x^2 + 3y^2 + 6xy - 3x^2 - 3y^2 + 6xy$$

$$= 12xy$$

$$\frac{\partial u}{\partial x} = 6xy --------(6)$$

$Consider\ f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \left(3x^2 - 3y^2\right) + i6xy\left[by\ \ (5)\&(6)\right]$

By Milne's Thomson method replace x by z and y by 0

$f'(z) = 3z^2$

int *egrat*

$f(z) = z^3 + c$

5) Find an analytic function f(z)-u+iv, given that u+v= $\frac{1}{r^2}\left[\cos 2\theta - \sin 2\theta\right], r \neq 0$

$Solution: u + v = \frac{1}{r^2}\left[\cos 2\theta - \sin 2\theta\right] - - - - - (1)$

Differentiate (1) *w.r.t. r*

$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r} = -\frac{2}{r^3}\left[\cos 2\theta - \sin 2\theta\right] - - - - - (2)$

Differentiate (1) *w.r.t.*θ

$\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial \theta} = \frac{2}{r^2}\left[-2\sin 2\theta - 2\cos 2\theta\right] - - - - - (3)$

By C-R Equations

$\left.\begin{array}{l} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \end{array}\right\}$ in LHS of (3)

$$-r\frac{\partial v}{\partial r}+r\frac{\partial u}{\partial r}=\frac{-2}{r^2}[\sin 2\theta+\cos 2\theta]$$

$$\frac{\partial u}{\partial r}-\frac{\partial v}{\partial r}=\frac{-2}{r^3}[\sin 2\theta+\cos 2\theta]------(4)$$

Consider

$$\frac{\partial u}{\partial r}+\frac{\partial v}{\partial r}=\frac{-2}{r^3}[\cos 2\theta-\sin 2\theta]$$

$$\frac{\partial u}{\partial r}-\frac{\partial v}{\partial r}=\frac{-2}{r^3}[\cos 2\theta+\sin 2\theta]$$

$$2\frac{\partial u}{\partial r}=\frac{-2}{r^3}[2\cos 2\theta]$$

$$\frac{\partial u}{\partial r}=\frac{-2}{r^3}\cos 2\theta----------------(5)$$

Subtract (3)-(4) we get

$$2\frac{\partial u}{\partial r}=-\frac{2}{r^3}[-2\sin 2\theta]$$

$$\frac{\partial u}{\partial r}=\frac{2}{r^3}\sin 2\theta----------(6)$$

Consider $f'(z)=e^{-i\theta}\left[\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right]$

$$f'(z)=e^{-i\theta}\left[-\frac{2}{r^3}\cos 2\theta+i\frac{2}{r^3}\sin 2\theta\right]$$

By Milne's Thomson method replace r by z and θ by 0

$$f'(z)=-\frac{2}{r^3}$$

int*egrate*

$$f(z)=-2\left(-\frac{1}{2z^2}\right)+c$$

$$f(z)=\frac{1}{z^2}+c$$

6) Show that $u = \left(r + \frac{1}{r}\right)\cos\theta$ is harmonic. find its harmonic conjugate and also corresponding analytic function.

Solution: Given $u = \left(r + \frac{1}{r}\right)\cos\theta$ --------(1)

we shall show that u is a solution of Laplace's equation in two variables in polar form.

i.e $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - 0$ ----------(2)

Differentiate (1) w.r.t. r

$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta$ ----------------(3)

Differentiate (3) w.r.t. r

$\frac{\partial^2 u}{\partial r^2} = +\frac{2}{r^3}\cos\theta$ -----------------(4)

Differentiate (1) w.r.t. θ

$\frac{\partial u}{\partial \theta} = \left(1 + \frac{1}{r}\right)(-\sin\theta)$ ----------------(5)

Differentiate (5) w.r.t. θ

$\frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{1}{r}\right)\cos\theta$ --------(6)

Consider

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2}{r^3}\cos\theta + \frac{1}{r}\left(1 - \frac{1}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{1}{r}\right)\cos\theta$$

$$= \frac{2}{r^3}\cos\theta + \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta - \frac{1}{r}\cos\theta - \frac{1}{r^3}\cos\theta$$

$$= \frac{2}{r^3}\cos\theta - \frac{2}{r^3}\cos\theta$$

$$= 0$$

$\therefore u$ is solution of equation(2)

Hence u is harmonic function.

Consider

$$f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] \text{-------}(7)$$

By C-R Equation $\dfrac{\partial u}{\partial \theta} = \text{-r}\dfrac{\partial v}{\partial r}$

$\therefore$ *replace* $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ *in* (7)

$$f'(z) = e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta - \frac{i}{r}\left(r + \frac{1}{r}\right)\sin\theta\right]$$

By Milne's Thomson method replace r by z and θ by 0

$$f'(z) = \left(1 - \frac{1}{z^2}\right) - i.o$$

$$f'(z) = \left(1 - \frac{1}{z^2}\right)$$

Integrate

$$f(z) = z + \frac{1}{z}$$

To find harmonic Conjugate

consider $u + iv = re^{i\theta} + \dfrac{1}{r}e^{-i\theta}$

$$u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating real and imaginary parts

$$\therefore u = \left(r + \frac{1}{r}\right)\cos\theta$$

$$v = \left(r - \frac{1}{r}\right)\sin\theta$$

which is required conjugate harmonic

7) If *f(z)* is a regular function of z show that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$

Solution:

$$\text{We have } f(z)=u+iv$$

$$\therefore |f(z)|=\sqrt{u^2+v^2} ---------(1)$$

$$|f(z)|^2=u^2+v^2 ---------(2)$$

$$\text{and } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\therefore |f'(z)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2}$$

$$|f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 -----(3)$$

Differentiate (2) w.r.t. *x*

$$\frac{\partial |f(z)|^2}{\partial x}=\frac{\partial}{\partial x}(u^2+v^2)$$

$$=2u\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial x}$$

Again differentiate w.r.t. *x*

$$\frac{\partial^2|f'(z)|^2}{\partial x^2}=2\left\{\frac{\partial}{\partial x}\left[u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}\right]\right\}$$

$$=2\left\{u\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}+v\frac{\partial^2 v}{\partial x^2}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial x}\right\}$$

$$2\left\{u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2+v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right\} ------(4)$$

Similarly Differentiate (2) w.r.t. y we get

$$\frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left\{u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right\} ------(5)$$

Adding (4) and (5) we get

$$\frac{\partial^2 |f(z)|^2}{\partial x^2} + \frac{\partial^2 |f(z)|^2}{\partial y^2} = 2\left\{u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\} ---(6)$$

w. k. t. if $f(z)=u+iv$ is regular or analytic function then real part u and imaginary part v satisfies Laplace equation in two variables or two dimensional Laplace equation.

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Using these on RHS of (6)

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right\}$$

By C-R Equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$= 2\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(-\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\}$$

$$= 2\left\{2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$= 4\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right\}$$

$$= 4|f'(z)|^2 \quad \text{[from (3)]}$$